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The Return of the Flowing Continuum

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ABSTRACT: After the arithmetization of the continuum in the 19th century, L.E.J. Brouwer, in his 1907 PhD thesis, brought back the intuitive continuum on the basis of his intuitionistic philosophy. The so-called ur-intuition (or intuition of time) yielded at the same time the natural numbers, and their derivatives, and the continuum. We show that intuitionistic mathematics, being a constructive mental activity of the subject, had to break with the traditional laws of logic, in particular with the principle of the excluded middle. We go into the properties of the continuum and the natural numbers, as spelled out by Brouwer. In particular we argue that the ur-intuition (embodied in the move of time) encompasses induction and recursion. The second installment of the intuitionistic revolution, beginning in 1918, resulted in an autonomous Brouwerian universe of mathematics, based on choice sequences and continuity.

Keywords: continuum, ur-intuition, choice sequence, induction, natural numbers, principle of the excluded middle, continuity principle, implication.

RESUME : **La fluidité du continu réhabilitée**. Après l'arithmétisation du continu au 19ème siècle, L.E.J. Brouwer, dans sa thèse de Doctorat en 1907, ramena le continu intuitif au sein de sa philosophie intuitioniste. Ce qu'il est convenu d'appeler ur-intuition (ou intuition du temps) permit l'appréhension conjointe des entiers naturels et du continu. Nous montrons que les mathématiques intuitionistes, en tant qu'activité mentale du sujet, furent contraintes de rompre avec certaines lois traditionnelles de la logique, en particulier le principe du tiers exclu. Nous examinons les propriétes du continu et des entiers naturels selon la vision brouwérienne. Nous établissons, en particulier, que la notion de ur-intuition (qui prend corps dans le glissement du temps) admet comme cas particulier l'induction et la récursion. Le deuxième acte de l'intuitionisme, amorcé en 1918, donna naissance à un univers brouwérien autonome au sein des mathématiques, univers fondé sur les suites de choix et le principe de continuité.

Mots clés : Continuum, ur-intuition, suites de choix, induction, entiers naturels, principe du tiers exclu, principe de continuité, implication.

> Die Eisdecke war in Schollen zerborsten, und jetztz ward das Element des Fließenden bald vollends Herr über das Feste.
>
> Herman Weyl

The nineteenth Century brought mathematics a triumph that finally settled the "scandal" of the continuum, or the line, of traditional geometry. Ever since

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the Greeks, the secrets of the line had been approached geometrically, through the postulated intersections of lines, circles, …. That technique had, however, never been sufficiently refined to clarify the intrinsic nature of the line. In particular it had so far not shed enough light on the old controversy "infinite divisibility" versus "atomicity". Those who adhered to the first option, let us call them for lack of a better name 'continuists', maintained that any portion of the line could be divided into smaller parts; the second group, the atomists, defended the view that the line was made up of smallest particles, called points; these atoms could not be subdivided any further. Both sides were represented by thinkers that were not them least in the world of philosophy and science. The most influential continuist had of course been Aristotle; he argued that between points there is always a (part of) a line, so that the line has to be infinitely divisible. The name of Democritus is best known for his atomistic theory of space and the continuum.

Improved insights into convergence of sequences, into least upper bounds of sets, continuity, and curves, finally allowed leading analysts such as Dedekind, Cantor, Weierstrass to put the theory of the continuum on a firm footing. Real numbers were defined on the basis of the rational numbers by means of Cauchy sequences (converging sequences), sequences of shrinking rational segment, or Dedekind cuts (least upper bounds of certain sets of rationals). As the continuum was thus reduced to the rational numbers, and eventually to the natural numbers, the process was often referred to as the arithmetization of the continuum (or the whole of analysis). Combining these techniques with the newly founded theory of sets of Cantor, one could finally and safely conclude that the real line, i.e. the continuum, was no more and no less than a set of mathematical objects. The continuum was thus made up of smallest sets, namely singletons of points, that could not be subdivided any further.

The atomistic conception of the continuum had in the end triumphed, and the mathematical community embraced the new foundation of analysis *en masse*, shelving the continuum dispute with other historically outdated matters. The process turned indeed out to be completely successful in explaining and handling the basic properties of the continuum and its functions. Thus at the end of the nineteenth century the problems of the foundations of analysis seemed settled once and for all.

One can imagine the surprise of the community, when in 1907 an unknown young Dutchman presented a novel foundation of the continuum that rejected the atomist credo. L.E.J. Brouwer defended in that year a doctoral dissertation in which he dealt with a mixed bag of mathematical topics; he solved a special case of 'Hilbert 5', the problem of founding Lie group theory without differentiability conditions, put forward a new conception of mathematics, including a foundation of the continuum, and he thoroughly inspected and criticized the existing foundational practices.

It is not easy to get a satisfactory understanding of his approach to continuum from the text of the dissertation as a consequence of an unfortunate decision of the PhD advisor, D.J. Korteweg. Korteweg was sincerely worried that Brouwer's philosophical – and somewhat mystical – elaborations of his ideas would not really please the science faculty, and so he insisted that certain parts should be removed from the text. The result was that some essential philosophical and heuristic explanations were not adopted in the final version. In order to put Brouwer's ideas into a proper perspective, it is useful to summarise briefly those missing parts, see van Stigt (1979).

Brouwer's point of departure was the activity of the human mind. Roughly speaking, the mathematical objects are the creation of the human being, traditionally called "the subject". The subject experiences a certain flow of sensations; these sensations may just come and go without any reaction of the subject, but at a certain point the subject may actively interfere as follows: the subject notes, or fixes his attention on a sensation, which is subsequently followed by another sensation. The first sensation is stored in memory at its passing away, being replaced by another sensation. The result is that the subject is aware of both the remembered and the present sensation. This is what Brouwer calls a "two-ity". Two-ities occur incessantly; at a following stage they are by an act of abstraction identified by the subject into the "empty two-ity". This phenomenon is the basis of Brouwer's genesis of mathematics and its objects. In later publications the procedure is discussed further, and it remains the cornerstone of the creation of mathematics. The above described act is also called *the move in time*, or *the falling apart of a moment of life*, see Brouwer (1912, 1929, 1949).

Here is an early formulation, as formulated in the rejected parts: "The ur-phenomenon is the intuition of time by itself, in which iteration, as 'thing in time, and one more thing', is possible, but in which also a sensation can fall apart into composing qualities, so that a single moment of life can be lived as a causal sequence of qualitative distinct things."

The dissertation itself contains a slight variation of the above, see Brouwer, (1907, p. 81), see also van Dalen (2005): "The ur-phenomenon is the intuition of time by itself, in which iteration, as 'thing in time, and one more thing', is possible, and on the basis of which moments of life fall apart as causal sequences of qualitative distinct things, that subsequently concentrate themselves in the intellect as not sensed, but observed mathematical causal sequences."

The notion of "ur-intuition of mathematics" appears quite early in the dissertation; it would in fact have benefited from an elucidation as the one above. As it is, it comes rather out of the blue. It is presented as "the substratum of all perception of change, which is divested of all quality, a unity of continuous and discrete, a possibility of the thinking together of several units, connected by a "between", which never exhausts itself by the interpolation of new units." Indeed, the formulation is more like a synopsis of earlier considerations than like a statement that can bear the full responsibility of explaining a new notion. A year later, at the International Mathematics Conference in Rome, Brouwer returned to the ur-intuition in somewhat clearer terms:

> "*Ur-intuition of two-ness (two-ity)*: The intuitions of the continuous and the discrete join here, as the second is thought not by itself, but under preservation of the recollection of the first. The first and the second are thus *kept together* and the intuition of the continuous consists in this keeping together (continere = keeping together). This mathematical ur-intuition is nothing but the contentless abstraction of the sensation of time. That is to say, the sensation of "fixed" and "floating" together, or of "remaining" and "changing" together."

As the reader will no doubt observe, there is a definite Aristotelian ring to this formulation. The flow from the first to the second yields the intermediate continuum, which cannot be exhausted "by the interpolation of new units". The most reasonable reading of this clause is that the new units form a set in the sense of Brouwer's dissertation; that is to say, a set that is the result of iteration of denumerable sets (ω-series). In particular one cannot exhaust the intermediate continuum by a denumerable set of points. This excludes, so to speak, the rationals as a counterexample. The important point here is that the natural numbers and the continuum come about at the same act. One cannot have one without the other. The identification of the continuum intuition and the time intuition is quite explicit here.

It should not come as a surprise that the act involved in the creation of a two-ity can be repeated so that another move in time (or the shift to the next sensation) creates a *three-ity* (say a triple) and then also the empty triple. The process does not halt at any point; the act of *free unfolding* (Brouwer's terminology) yields the unending sequence of natural numbers, 1, 2, 3, 4, 5, ... One might object that the sequence, as motivated here, starts with 2; but one should take into account that once 2 (or the *pair*) has been created, the first component can be recognised, so that the number 1 is also obtained.

Before passing on to the continuum, let us have another look at the natural numbers. A not unreasonable objection to Brouwer's creation of the natural numbers would be that at any given stage only a limited part of the number sequence could have been actually constructed. In technical terms: one gets the individual numbers, but not the whole sequence. Here the 'free unfolding' is essential. At this point, indeed, Brouwer's dissertation is less than clear. That is to say, without the missing section, and without an explicit reference to the genesis of the number sequence, one might wonder where the extra step is made. The same point comes up with respect to the principle of complete induction.

Here is a relevant quote from a letter of Brouwer to the Utrecht mathematician Jan de Vries: "I replace the 'axiom of complete induction' by the mathematical construction-act of complete induction, and show that it is nothing new after the intuition of time." (undated, probably February/March 1907). The message here is that the *construction act of complete induction* yields the natural numbers as a whole, albeit not as a completed set such as e.g. {1.5.9}, but as a potential totality. Indeed, in modern language, one could say that the intuition of the natural number sequence automatically gives us the recursor operator. In the dissertation and various other places, Brouwer uses the term "the mathematical intuition *and so on*" (discussion of Russell), and at another spot he comments on Dedekind: "Dedekind's system has no meaning; a logical meaning would require a consistency proof, which Dedekind does not give either, then he would have to appeal to the intuition of 'and so on'." In later publications Brouwer specifies the generation of the number sequence as caused by the "self-unfolding of the act of the intellect". The paper *Consciousness, Philosophy, and Mathematics* spells it out: "Mathematics comes into being, when the two-ity created by a move of time is divested of all quality by the subject, and when the remaining empty form of the common substratum of all two-ities, as a basic intuition of mathematics is left to an unlimited unfolding, creating new mathematical entities in the shape of *predeterminately or more or less freely proceeding infinite sequences* of mathematical entities previously acquired, …" We note that thus Brouwer accepts the "and-so-on" intuition, or the number sequence as immediately given by intuition.

In his *Intuitionism and Formalism* Brouwer boldly asserts, "This intuition of two-oneness, the basal intuition of mathematics, creates not only the numbers one and two, but also all finite ordinal numbers, inasmuch as one of the elements of the two-oneness may be thought of as a new two-oneness, which process may be repeated indefinitely; this gives rise to the smallest infinite ordinal ω."

Given the intuitionistic reading of the logical connectives (according to the 'proof = construction' interpretation) one immediately gets the *principle of complete induction* as a corollary. Let us have a quick look at the matter. The argument runs roughly as follows: given $A(1)$ and $\forall n(A(n)) \rightarrow (A(n+1))$, we want to show $\forall n(A(n))$. 'Show' means for a constructivist 'present a proof', where we have to keep in mind that already in 1907 Brouwer was aware that proofs are constructions; he spoke of "erecting mathematical buildings" and "fitting buildings into other buildings". In modern terms this would be read as "constructing mathematical structures" and "constructing a structure on the basis of (out of) another structure". It is quite clear that he knew how proof-constructions for implication, universally and existentially quantified statements were to be made. The cases of conjunction and disjunction were tacitly understood. So – returning to the matter of induction – we may assume that there is a proof a_1 of $A(1)$; notation $a_1 : A(1)$. Now a proof for $\forall n(A(n)) \rightarrow (A(n+1))$ is a construction c that, for any given n and proof $a : A(n)$, yields a proof $c(n, a) : A(n+1)$.

So $a_2 = c(1, a_1)$ and $a_2 : A(2)$, and $a_3 = c(2, a_2)$ and $a_3 : A(3)$, …. Hence parallel to the construction of the natural numbers we obtain the (potentially infinite) sequence of proofs $a_1, a_2, a_3, \ldots$, i.e. a proof for $\forall n(A(n))$.[1]

Now back to the continuum: the genesis of the pair in the move in time creates simultaneously the "in between", the continuous move from one sensation to the next. This extra object and the pair arise inseparably at the same act, as formulated by Brouwer in his *ur-intuition*. The continuum escapes enumeration by points or atoms; in Brouwer's words: it is not a *set* but a "matrix of points that can be thought together". The continuum in this framework of "move in time" is with good reason identified with the passage/course of time. To formulate it more precisely, time is created in the act of shift from one sensation to the next, as well as the temporal discrete two-ity.

Whatever conclusion one may want to draw from the basic acts of the subject, it is clear that time (interchangeable with the continuum) is an a-priori notion; in contrast, space – both on account of the non-euclidean geometries of the nineteenth century, and of the new physics of relativity – had lost its claim on a-priority. The basis of mathematics thus is time, and space forms are derived from it. The continuous, flowing character of the continuum was in Brouwer's view inherent to its creation, to the move in time that in its transition from sensation to sensation did not allow for 'gaps'. For a proof of the non-atomicity more, however, was needed. Tools for this purpose were not available at the time of the dissertation; only after Brouwer's successful handling of choice phenomena progress became possible.

Already in 1905 Brouwer had come out in favour of the free will – "You recognise your "Free Will", in so far as it was free to withdraw itself from the world, in which there was causality, and then remains free, and yet only then has a really determined Direction, which it reversibly follows in freedom" – and free will was at the basis of the existence and justification of so-called *choice sequences*, which were "more or less freely proceeding sequences" of, say, natural numbers. In the dissertation hints at choice sequences can be found; the fact that the continuum cannot be exhausted by "known points" (we would now say "lawlike points") more or less forced the notion of "unknown points" on Brouwer. The same problem had, by the way, worried Borel, who also had come to look at choice sequences. Reading between the lines, and taking into account certain later comments, e.g. in a letter from Brouwer to Fraenkel, 12.I.1927, (van Dalen, 2000), it is not far-fetched to recognise choice sequences in the continuum; there is a passage that contains a spread-like picture and that applies techniques in his analysis if the notion of perfect sets, that reappear in later papers dealing with spreads and Cantor's fundamental theorem. It took him however till 1916/17 to find the key to a coherent view and use of choice sequences.

Without going further into the choice phenomena, we note that Brouwer returned to the standard techniques for generating the reals. Real numbers are

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[1]In systems with an explicit recursor, one can write down a term for the proof-construction given by the sequence.

again generated by sequences of rationals (respectively by sequences of shrinking rational segments), but now allowing choice sequences of rationals that satisfy the Cauchy condition. At this point the unfinished nature of infinite sequences received a second potential feature, one that added uncertainty on a different scale. Lawlike sequences, such as for example the sequence of primes, do not lie before us like a finished totality, but there is a strong argument for accepting them as well-determined individual objects that are completely captured by a single law or description that can be applied by any individual, and that can be communicated. A choice sequence on the other hand is highly undetermined, in the sense that at any stage only an initial segment is known, and future elements have a certain degree of freedom, ranging from completely unrestricted to 'given by a law'. It is this peculiar feature that violates the demands of objectivity; the choice sequence generated by a subject can in general not be shared by other subjects. This lack of objectivity in the notion of choice sequence has been the single most questioned and distrusted issue of intuitionism. Be that as it may, Brouwer had found a magic key to handle ordinary mathematical objects; choice sequences could be handled just like traditional sequences.

The basic principle that allowed Brouwer to develop his program was the so-called *continuity principle*; it was the key to handling quantifiers over choice sequences. Basically it came down to something like 'well-determined discrete properties of a choice sequence depend on a finite initial segment of such a sequence.' For precise formulations and a conceptual justification see (van Atten & van Dalen, 2002).

Hermann Weyl, who adopted choice sequences in his second constructive program, systematically viewed a choice sequence determining a real as a metaphor for all future possible continuations of initial segments, say an approximation given by the initial sequence plus the Cauchy modulus, (van Atten, van Dalen, Tieszen, 2002). His interpretation of choice sequence quantifiers clearly reflected this view. For Brouwer, on the other hand, a choice sequence was as much an individual, as for example the sequence (2^{-n}). One thing about choice sequences becomes apparent: they have an individuality that is only vaguely reflected in the way they can be discerned in the mathematical universe. In particular we must admit a certain leeway to real numbers, when viewed as points on the real line, or continuum. Since at a certain stage the information about a number is incomplete, we cannot with certainty locate it with respect to other numbers. Its place on the continuum is somewhat blurred, so to speak. Numbers cannot be precisely pinpointed at the line; at whichever stage we are, they resemble more small spots or intervals than atoms that carry all the information about their location, which cannot any further be refined. A point in the intuitionistic universe has, in phenomenological terms, a definite horizon that holds the future for the point, but that does not give the "infinite precision" that is typical for 2/7 or π. Whereas the classical continuum (and mathematical universe) is static, due to the force of principle of the excluded middle, the intuitionistic continuum (and universe) has a dynamic nature. This is indeed what is dynamical with respect to Brouwer's universe.

Having reached the point where one could say that Brouwer too had embraced the reductionist view of the continuum, i.e. that the continuum is a by-product of the theory of number sequences (Baire space), we should ask ourselves if this is the bankruptcy of the ur-intuition. From a conceptual point of view this must be denied; after all, if one accepts the generation of real numbers by means of sequences of rationals, then one also accepts the underlying fact that there is some idea or intuition that precedes the resulting structure, just as the vibrating string precedes the differential equation that represents it. Before one can codify a notion by available techniques, one must have an intuition of the notion. So the intuition of the continuum must have preceded its representation by Cauchy sequences. In Brouwer's *Die Struktur des Kontinuums* this is explicitly mentioned: "so that the conception of the continuum, mentioned in the beginning, as pure intuition after Kant and Schopenhauer, remains correct". In a handwritten private note Brouwer remarked, "In the continuum lecture, at the end of section I, add that the continuum is after all *given immediately* by the ur-intuition, just as with Kant and Schopenhauer." In later publications the genesis of the continuum has been stressed to a smaller extent, but also there the representation of the continuum presupposes the continuum.

A topic that figured fairly prominent in the dissertation, and that completely disappeared after the "new intuitionism" was conceived, is the continuum hypothesis. In the dissertation Brouwer claimed to have solved it. He had analyzed the possible sets on the basis of his first program; these sets were those that could be constructed stepwise from the basic objects. As a consequence he recognized only finite and denumerable sets, for example the countable ordinals, but not the second number class as a whole. The continuum was accepted as a matrix, given immediately by intuition; it could however not be constructed as a set of individual elements. Hence it was only as a generalized intuitive mathematical construct, that it entered into mathematics. The answer to the continuum hypothesis was thus affirmative. Brouwer retracted this position soon after he realized the consequences of the notion of choice sequence. And from then on the matter lost all interest, as the wealth of possible sets defeated any classification on the basis of cardinality.

The best known, and for some time most controversial consequence of Brouwer's unusual continuum (in combination with the introduction of a constructive logic) was the *Continuity Theorem*: every real function is (locally uniformly) continuous. An immediate corollary was the theorem that strikingly demonstrated the non-atomicity of the continuum: *the indecomposability of the continuum* – if the continuum is the disjoint union of A and B then one of the two is empty. Thus the equality relation is obviously not decidable, for otherwise $\forall x\,(x = 0 \vee x \neq 0)$, and we would have $\mathbf{R} = \{0\} \cup \{x \in \mathbf{R} : x \neq 0\}$. This, by itself, makes an atomistic continuum an illusion; large groups of choice reals may be virtually indistinguishable, and hence one cannot be certain whether such a group can be subdivided.

It is worthwhile to note that the indecomposability of the continuum was Brouwer's first strong refutation of the principle of the excluded middle. Under an extra, and not extravagant principle – the *Kripke-Brouwer schema* – it can be shown that $\neg\forall x \in \mathbf{R}\ (x = 0 \vee x \neq 0)$. The continuum of Brouwer shows a strong 'syrup-like' behaviour; one cannot cut up the continuum into two parts, because in the act of cutting the status of a large number of reals is left open, e.g. suppose one wants to cut the continuum at the origin, then there are lots of points for which $a > 0$, $a = 0$ or $a < 0$ is unknown.

One can go even a bit further; one might try to split the continuum by perforating it, much as the postal service does by perforating sheets of stamps. However, even that does not yield a partitioning, as $\mathbf{R} \setminus \{0\}$ is indecomposable as well. Even the sets of irrationals, and of non-non-rational turn out to be indecomposable, cf. Van Dalen (1999). Thus Brouwer's continuum is extremely tight; it is connected to a degree that greatly surpasses the topological connectedness.

Looking at mathematics as a whole, we may say that Brouwer has given mathematics a quite specific intuitionistic universe with properties strikingly different from the traditional classical universe (nowadays mostly dressed in the garb of ZFC). One could not possibly have obtained these modifications without adapting the underlying logic; the principle of the excluded middle is such a powerful restriction that the freedom of creating alternative universes is severely restricted. So one might wonder if the abundance of alternative universes could be the consequence of logic alone. This definitely is not the case; one needs alternative principles in actual mathematics.

Some philosophers have jokingly or scathingly spoken of deviant logics; that, however, seems to be a gross misunderstanding of the value and force of conceptual analysis of the notions concerned. Brouwer's logic of "proof = construction", codified by Heyting and Kolmogorov, and subsequently given a precise treatment in Martin-Löf's type-theory, and other type theories, was definitely not just a variation on a familiar theme, but the outcome of a conceptual analysis of the meaning of the logical notions. In short, it is the combination of logical and mathematical principles that determines the nature of our mathematical universe, and the surprising fact is that genuine conceptual analysis leads fairly easily to insights that have unexpected consequences for the continuum. One may, of course, wonder what is the secret of success of analysis when dealing with alternative universes, when compared to first-order arithmetic. The fact that even in intuitionistic logic equality of numbers is decidable stands in the way of easy results. Here, apparently, the fact that the objects are so similar, not to say identical, in the corresponding settings, seems to put a heavier burden on the logic. The one mathematical principle that yields strong and unexpected results is Church's Thesis, which in a constructive setting reads "all number theoretic functions are recursive", or even "all $\forall\exists$" combinations are witnessed by recursive functions". As McCarty showed, it entails, among other things, that there are no non-standard models (McCarthy, 1988).

Finally I would like to call attention to the fact that alternative universes for mathematics are the staple diet of model theory, in particular in topos theory. The possibilities to obtain unusual universes are abundant. For example, there are universes in which all real functions are not just continuous, but actually differentiable. The question, whether the continuum is after all continuous or atomic seems thus a highly delicate one.

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